

Exponents for three-dimensional simultaneous Diophantine approximations

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Abstract

Let $\Theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$. Suppose that $1, \theta_1, \theta_2, \theta_3$ are linearly independent over \mathbb{Z} . For Diophantine exponents

$$\alpha(\Theta) = \sup\{\gamma > 0 : \limsup_{t \rightarrow +\infty} t^\gamma \psi_\Theta(t) < +\infty\},$$

$$\beta(\Theta) = \sup\{\gamma > 0 : \liminf_{t \rightarrow +\infty} t^\gamma \psi_\Theta(t) < +\infty\}$$

we prove

$$\beta(\Theta) \geq \frac{1}{2} \left(\frac{\alpha(\Theta)}{1 - \alpha(\Theta)} + \sqrt{\left(\frac{\alpha(\Theta)}{1 - \alpha(\Theta)} \right)^2 + \frac{4\alpha(\Theta)}{1 - \alpha(\Theta)}} \right) \alpha(\Theta)$$

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1. Diophantine exponents.

Let $\Theta = (\theta_1, \dots, \theta_n)$ be a real vector. We deal with the function

$$\psi_\Theta(t) = \min_{x \leq t} \max_{1 \leq i \leq n} \|\theta_i x\|.$$

Here the minimum is taken over positive integers x and $\|\cdot\|$ stands for the distance to the nearest integer.

Suppose that at least one of the numbers $\theta_1, \dots, \theta_n$ is irrational. Then $\psi_\Theta(t) > 0$ for all $t \geq 1$. The *uniform* Diophantine exponent $\alpha(\Theta)$ is defined as the supremum of the set

$$\{\gamma > 0 : \limsup_{t \rightarrow +\infty} t^\gamma \psi_\Theta(t) < +\infty\},$$

It is a well-known fact that for all Θ one has

$$\frac{1}{n} \leq \alpha(\Theta) \leq 1.$$

The *ordinary* Diophantine exponent $\beta(\Theta)$ is defined as the supremum of the set

$$\{\gamma > 0 : \liminf_{t \rightarrow +\infty} t^\gamma \psi_\Theta(t) < +\infty\}.$$

Obviously

$$\beta(\Theta) \geq \alpha(\Theta). \tag{1}$$

2. Functions.

For each $\alpha \in [\frac{1}{3}, 1)$, define

$$g_1(\alpha) = \frac{\alpha}{1 - \alpha}$$

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and

$$g_2(\alpha) = \frac{\alpha(1-\alpha) + \sqrt{\alpha(\alpha^3 + 6\alpha^2 - 7\alpha + 4)}}{2(2\alpha^2 - 2\alpha + 1)}.$$

The value $g_2(\alpha)$ is the largest root of the equation

$$(2\alpha^2 - 2\alpha + 1)x^2 + \alpha(\alpha - 1)x - \alpha = 0.$$

Note that

$$g_2(1/3) = g_2(1) = 1,$$

and for $1/3 < \alpha < 1$ one has $g_2(\alpha) > 1$. Let α_0 be the unique real root of the equation

$$x^3 - x^2 + 2x - 1 = 0.$$

In the interval $1/3 < \alpha < \alpha_0$ one has

$$g_2(\alpha) > \max(1, g_1(\alpha)). \quad (2)$$

In the interval $\alpha_0 \leq \alpha < 1$ we see that

$$g_2(\alpha) \leq g_1(\alpha).$$

We define one more function. Put

$$g_3(\alpha) = \frac{1}{2} \left(\frac{\alpha}{1-\alpha} + \sqrt{\left(\frac{\alpha}{1-\alpha} \right)^2 + \frac{4\alpha}{1-\alpha}} \right). \quad (3)$$

Simple calculation shows that

$$g_3(\alpha) > \max(g_1(\alpha), g_2(\alpha)) \quad \forall \alpha \in \left(\frac{1}{3}, 1 \right). \quad (4)$$

3. Jarník's result.

In a fundamental paper [1] V. Jarník proved the following theorem.

Theorem 1.

Let $\psi(t)$ be a continuous function in t , decreasing to zero as $t \rightarrow +\infty$. Suppose that the function $t\psi(t)$ increases to infinity as $t \rightarrow +\infty$. Let $\rho(t)$ be the inverse function to the function $t\psi(t)$. Put

$$\varphi^{[\psi]}(t) = \psi \left(\rho \left(\frac{1}{6\psi(t)} \right) \right).$$

Suppose that $n \geq 2$ and among numbers $\theta_1, \dots, \theta_n$ there exist at least two numbers which, together with 1, are linearly independent over \mathbb{Z} . Suppose that

$$\psi_{\Theta}(t) \leq \psi(t)$$

for all t large enough. Then there exist infinitely many integers x such that

$$\max_{1 \leq j \leq n} \|x\theta_j\| \leq \varphi^{[\psi]}(x).$$

The next Jarník's result on Diophantine exponents is an obvious corollary of Theorem 1.

Theorem 2.

Suppose that $n \geq 2$ and among numbers $\theta_1, \dots, \theta_n$ there exist at least two numbers which, together with 1, are linearly independent over \mathbb{Z} . Then

$$\beta(\Theta) \geq \alpha(\Theta)g_1(\alpha(\Theta)).$$

To obtain Theorem 2 from Theorem 1 one takes $\psi(t) = t^{-\alpha}$ with $\alpha < \alpha(\Theta)$.

From the other hand V. Jarník [1] proved that there exists a collection of numbers $\Theta = (\theta_1, \dots, \theta_n)$ such that $1, \theta_1, \dots, \theta_n$ are linearly independent over \mathbb{Z} and

$$\beta(\Theta) < \frac{\alpha(\Theta)}{1 - \alpha(\Theta)}.$$

In the case $n = 2$ the lower bound of Jarník's Theorem 2 is optimal. The following result was proved by M. Laurent [2].

Theorem 3.

For any $\alpha, \beta > 0$ satisfying

$$\frac{1}{2} \leq \alpha \leq 1, \quad \beta \geq \alpha g_1(\alpha)$$

there exists a vector $\Theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ such that

$$\alpha(\Theta) = \alpha, \quad \beta(\Theta) = \beta.$$

This result is a corollary of a general theorem concerning four two-dimensional Diophantine exponents.

Note that in the case $n \geq 3$ the bound of Theorem 2 in the range $\frac{1}{n} \leq \alpha < \frac{1}{2}$ is weaker than the trivial bound (1).

N. Moshchevitin [3] (see also [4], Section 5.2) improved Jarník's result in the case $n = 3$ and for $\alpha \in (\frac{1}{3}, \alpha_0)$. He obtained the following

Theorem 4. Suppose that $m = 1, n = 3$ and the collection $\Theta = (\theta_1, \theta_2, \theta_3)$ consists of numbers which, together with 1, are linearly independent over \mathbb{Z} . Then

$$\beta(\Theta) \geq \alpha(\Theta)g_2(\alpha(\Theta)).$$

In the case $n = 3$, Theorems 2 and 4 together give an estimate which is better than the trivial estimate (1) for all admissible values of $\alpha(\Theta)$.

4. New result.

In this paper we give a new lower bound for $\beta(\Theta)$ in terms of $\alpha(\Theta)$. From (4) it follows that this bound is stronger than all previous bounds (Theorems 2 and 4) for all admissible values of $\alpha(\Theta)$.

Theorem 5.

Suppose that $m = 1, n = 3$ and the vector $\Theta = (\theta_1, \theta_2, \theta_3)$ consists of numbers linearly independent, together with 1, over \mathbb{Z} . Then

$$\beta(\Theta) \geq \alpha(\Theta)g_3(\alpha(\Theta)).$$

Sections 5,6,7 below contains auxiliary results. Theorem 5 is proved in Section 8.

5. Best approximations.

For each integer x , put

$$\zeta(x) = \max_{1 \leq j \leq n} \|\theta_j x\|.$$

A positive integer x is said to be a *best approximation* if

$$\zeta(x) = \min_{x'} \zeta(x'),$$

where the minimum is taken over all $x' \in \mathbb{Z}$ such that

$$0 < x' \leq x.$$

Consider the case when all numbers 1 and θ_j , $1 \leq j \leq n$ are linearly independent over \mathbb{Z} . Then all best approximations lead to sequences

$$x_1 < x_2 < \dots < x_\nu < x_{\nu+1} < \dots ,$$

$$\zeta(x_1) > \zeta(x_2) > \dots > \zeta(x_\nu) > \zeta(x_{\nu+1}) > \dots .$$

We use the notation

$$\zeta_\nu = \zeta(x_\nu).$$

Choose $y_{1,\nu}, \dots, y_{n,\nu} \in \mathbb{Z}$ so that

$$\|\theta_j x_\nu\| = |\theta_j \mathbf{x}_\nu - y_{j,\nu}|.$$

We define

$$\mathbf{z}_\nu = (x_\nu, y_{1,\nu}, \dots, y_{n,\nu}) \in \mathbb{Z}^{n+1}.$$

If $\psi(t)$ is a continuous function decreasing to 0 as $t \rightarrow \infty$, with

$$\psi_\Theta(t) \leq \psi(t)$$

then one easily sees that

$$\zeta_\nu \leq \psi(x_{\nu+1}). \quad (5)$$

Some useful fact about best approximations can be found in [4].

6. Two-dimensional subspaces.

Lemma 1. *Suppose that all vectors of the best approximations \mathbf{z}_l , $\nu \leq l \leq k$ lie in a certain two-dimensional linear subspace $\pi \subset \mathbb{R}^4$. Consider two-dimensional lattice $\Lambda = \pi \cap \mathbb{Z}^4$ with two-dimensional fundamental volume $\det \Lambda$. Then for all l from the interval $\nu \leq l \leq k-1$ one has*

$$C_1 \det \Lambda \leq \zeta_l x_{l+1} \leq 2 \det \Lambda. \quad (6)$$

where $C_1 = \left(2 \sqrt{3 \left(1 + (|\theta_1| + \frac{1}{2})^2 + (|\theta_2| + \frac{1}{2})^2 + (|\theta_3| + \frac{1}{2})^2 \right)} \right)^{-1}$. In particular,

$$\det \Lambda \geq \frac{\min(\zeta_\nu x_{\nu+1}, \zeta_{k-1} x_k)}{2}. \quad (7)$$

Proof. The parallelepiped

$$\Omega_l = \{ \mathbf{z} = (x, y_1, y_2, y_3) : |x| < x_{l+1}, \max_{1 \leq j \leq 3} |\theta_j x - y_j| < \zeta_l \}$$

has no non-zero integer points inside for every l . Consider two-dimensional $\mathbf{0}$ -symmetric convex body

$$\Xi_l = \Omega_l \cap \pi.$$

One can see that the two-dimensional Lebesgue measure $\mu(\Xi_l)$ of Ξ_l admit the following lower and upper bounds:

$$2\zeta_l x_{l+1} \leq \mu(\Xi_l) \leq 4\sqrt{3\left(1 + \left(|\theta_1| + \frac{1}{2}\right)^2 + \left(|\theta_2| + \frac{1}{2}\right)^2 + \left(|\theta_3| + \frac{1}{2}\right)^2\right)} \zeta_l x_{l+1}. \quad (8)$$

We see that there is no non-zero points of Λ inside Ξ_l and that there are two linearly independent points $\mathbf{z}_l, \mathbf{z}_{l+1} \in \Lambda$ on the boundary of Ξ_l . So obviously

$$2 \det \Lambda \leq \mu(\Xi_l). \quad (9)$$

From the Minkowski convex body theorem it follows that

$$\mu(\Xi_l) \leq 4 \det \Lambda. \quad (10)$$

Now (6) follows from (8,9,10). Lemma is proved. \square

7. Three-dimensional subspaces.

Consider three consecutive best approximation vectors $\mathbf{z}_{l-1}, \mathbf{z}_l, \mathbf{z}_{l+1}$. Suppose that these vectors are linearly independent. Consider the three-dimensional linear subspace

$$\Pi_l = \text{span}(\mathbf{z}_{l-1}, \mathbf{z}_l, \mathbf{z}_{l+1}).$$

Consider the lattice

$$\Gamma_l = \Pi_l \cap \mathbb{Z}^4$$

with the fundamental volume $\det \Gamma_l$. Let Δ be three-dimensional volume of the three-dimensional simplex \mathcal{S} with vertices $\mathbf{0}, \mathbf{z}_{l-1}, \mathbf{z}_l, \mathbf{z}_{l+1}$. We see that

$$\Delta \geq \frac{\det \Gamma_l}{6}. \quad (11)$$

Consider determinants

$$\Delta_1 = - \begin{vmatrix} x_{l-1} & y_{2,l-1} & y_{3,l-1} \\ x_l & y_{2,l} & y_{3,l} \\ x_{l+1} & y_{2,l+1} & y_{3,l+1} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} x_{l-1} & y_{1,l-1} & y_{3,l-1} \\ x_l & y_{1,l} & y_{3,l} \\ x_{l+1} & y_{1,l+1} & y_{3,l+1} \end{vmatrix}, \quad \Delta_3 = - \begin{vmatrix} x_{l-1} & y_{1,l-1} & y_{2,l-1} \\ x_l & y_{1,l} & y_{2,l} \\ x_{l+1} & y_{1,l+1} & y_{2,l+1} \end{vmatrix}. \quad (12)$$

Absolute values of these determinants are equal to three-dimensional volumes of projections of the simplex \mathcal{S} onto three-dimensional coordinate subspaces ($\{y_1 = 0\}, \{y_2 = 0\}$ and $\{y_3 = 0\}$ respectively) multiplied by 6.

Note that for $j = 1, 2, 3$ one has

$$|\Delta_j| \leq 6\zeta_{l-1}\zeta_l x_{l+1}. \quad (13)$$

Lemma 2. *Among determinants (12) there exist a determinant with absolute value $\geq C_2 \Delta$, where $C_2 = 2/(2 + \max_{1 \leq i \leq 3} |\theta_i|)$.*

Proof.

Consider the determinant

$$\Delta_0 = \begin{vmatrix} y_{1,l-1} & y_{2,l-1} & y_{3,l-1} \\ y_{1,l} & y_{2,l} & y_{3,l} \\ y_{1,l+1} & y_{2,l+1} & y_{3,l+1} \end{vmatrix}$$

and the vector

$$\mathbf{w} = (\Delta_0, \Delta_1, \Delta_2, \Delta_3) \in \mathbb{Z}^4.$$

We see that \mathbf{w} is orthogonal to the subspace Π_l , that is

$$\Delta_0 x_j + \Delta_1 y_{1,j} + \Delta_2 y_{2,j} + \Delta_3 y_{3,j} = 0, \quad j = l-1, l, l+1.$$

So

$$\Delta_0 = - \sum_{i=1}^3 \Delta_i \frac{y_{i,l}}{x_l} = - \sum_{i=1}^3 \Delta_i \left(\frac{y_{i,l}}{x_l} - \theta_i \right) - \sum_{i=1}^3 \Delta_i \theta_i.$$

As $\left| \frac{y_{i,l}}{x_l} - \theta_i \right| \leq 1$ we see that

$$|\Delta_0| \leq (1 + \max_{1 \leq i \leq 3} |\theta_i|) (|\Delta_1| + |\Delta_2| + |\Delta_3|). \quad (14)$$

But

$$36\Delta^2 = \Delta_0^2 + \Delta_1^2 + \Delta_2^2 + \Delta_3^2. \quad (15)$$

From (14,15) we deduce the inequality

$$\Delta \leq \frac{1}{6} (2 + \max_{1 \leq i \leq 3} |\theta_i|) (|\Delta_1| + |\Delta_2| + |\Delta_3|),$$

and the lemma follows. \square

8. Proof of Theorem 5.

Take $\alpha < \alpha(\Theta)$. Then

$$\zeta_l \leq x_{l+1}^{-\alpha} \quad (16)$$

for all l large enough.

Consider best approximation vectors $\mathbf{z}_\nu = (x_\nu, y_{1,\nu}, y_{2,\nu}, y_{3,\nu})$. From the condition that numbers $1, \theta_1, \theta_2, \theta_3$ are linearly independent over \mathbb{Z} we see that there exist infinitely many pairs of indices $\nu < k, \nu \rightarrow +\infty$ such that

- both triples

$$\mathbf{z}_{\nu-1}, \mathbf{z}_\nu, \mathbf{z}_{\nu+1}; \quad \mathbf{z}_{k-1}, \mathbf{z}_k, \mathbf{z}_{k+1}$$

consist of linearly independent vectors;

- there exists a two-dimensional linear subspace π such that

$$\mathbf{z}_l \in \pi, \quad \nu \leq l \leq k; \quad \mathbf{z}_{\nu-1} \notin \pi, \quad \mathbf{z}_{k+1} \notin \pi;$$

- the vectors

$$\mathbf{z}_{\nu-1}, \mathbf{z}_\nu, \mathbf{z}_k, \mathbf{z}_{k+1}$$

are linearly independent.

Consider the two-dimensional lattice

$$\Lambda = \pi \cap \mathbb{Z}^4$$

By Lemma 1, its two-dimensional fundamental volume $\det \Lambda$ satisfies

$$\det \Lambda \asymp_{\Theta} \zeta_\nu x_{\nu+1} \asymp_{\Theta} \zeta_{k-1} x_k. \quad (17)$$

Consider the two dimensional orthogonal complement π^\perp to π and the lattice

$$\Lambda^\perp = \pi^\perp \cap \mathbb{Z}^4.$$

It is well-known that

$$\det \Lambda^\perp = \det \Lambda. \quad (18)$$

Consider the lattices

$$\Gamma_\nu = (\text{span}(\mathbf{z}_{\nu-1}, \mathbf{z}_\nu, \mathbf{z}_{\nu+1})) \cap \mathbb{Z}^4, \quad \Gamma_k = (\text{span}(\mathbf{z}_{k-1}, \mathbf{z}_k, \mathbf{z}_{k+1})) \cap \mathbb{Z}^4.$$

and primitive integer vectors $\mathbf{w}_\nu, \mathbf{w}_k \in \mathbb{Z}^4$ which are orthogonal to $\Pi_\nu = \text{span}(\mathbf{z}_{\nu-1}, \mathbf{z}_\nu, \mathbf{z}_{\nu+1})$, $\Pi_k = \text{span}(\mathbf{z}_{k-1}, \mathbf{z}_k, \mathbf{z}_{k+1})$ respectively. Obviously

$$\mathbf{w}_\nu, \mathbf{w}_k \in \Lambda^\perp.$$

Put

$$b = \frac{1}{2} \left(-\frac{\alpha}{1-\alpha} + \sqrt{\left(\frac{\alpha}{1-\alpha} \right)^2 + \frac{4\alpha}{1-\alpha}} \right) \in (0, 1), \quad a = 1 - b,$$

so

$$\frac{\alpha}{1-\alpha} + b = g_3(\alpha).$$

Then

$$\det \Lambda^\perp \leq |w_\nu| \cdot |w_k|,$$

where $|\cdot|$ stands for the Euclidean norm, and so we obtain that either

$$\det \Gamma_\nu = |\mathbf{w}_\nu| \geq (\det \Lambda^\perp)^a = (\det \Lambda)^a \quad (19)$$

or

$$\det \Gamma_k = |\mathbf{w}_k| \geq (\det \Lambda^\perp)^b = (\det \Lambda)^b \quad (20)$$

(using (18)).

If (19) holds then by Lemma 2, (13), (11) and (17) we see that

$$\zeta_{\nu-1} \zeta_\nu x_{\nu+1} \gg |\Delta_j| \gg_\Theta \det \Gamma_\nu \gg_\Theta (\det \Lambda)^a \gg (\zeta_\nu x_{\nu+1})^a$$

(here Δ_j is the determinant from Lemma 2 applied to the lattice $\Gamma = \Gamma_\nu$). From the definition of a and (16) we see that

$$x_{\nu+1} \gg_\Theta x_\nu^{g_3(\alpha)}.$$

We apply (16) again to obtain

$$\zeta_\nu \ll_\Theta x_\nu^{-\alpha g_3(\alpha)}.$$

If (20) holds then by Lemma 2, (13), (11) and (17) we see that

$$\zeta_{k-1} \zeta_k x_{k+1} \gg |\Delta_{j'}| \gg_\Theta \det \Gamma_k \gg_\Theta (\det \Lambda)^b \gg (\zeta_{k-1} x_k)^b$$

(here $\Delta_{j'}$ is the determinant from Lemma 2 applied to the lattice $\Gamma = \Gamma_k$). From the definition of b and (16) we see that

$$x_{k+1} \gg_\Theta x_k^{g_3(\alpha)}.$$

We apply (16) again to obtain

$$\zeta_k \ll_\Theta x_k^{-\alpha g_3(\alpha)}.$$

Theorem 5 is proved. \square

8. Acknowledgement and a remark.

The author thanks the anonymous referee for useful and important suggestions. Here we would like to note that the referee pointed out that it is possible to get a simpler proof of Theorem 5 by means of W.M. Schmidt's inequality on heights of rational subspaces (see [5]). For a rational subspace $U \subset \mathbb{R}^n$ its height $H(U)$ is defined as the co-volume of the lattice $U \cap \mathbb{Z}^n$. Schmidt shows that for any two rational subspaces $U, V \in \mathbb{R}^n$ one has

$$H(U \cap V)H(U + V) \ll_n H(U)H(V),$$

To prove our Theorem 5 one can use this inequality for

$$U = \text{span}(\mathbf{z}_{\nu-1}, \mathbf{z}_{\nu}), \quad V = \text{span}(\mathbf{z}_{\nu}, \mathbf{z}_{\nu+1})$$

and for

$$U' = V = \text{span}(\mathbf{z}_{k-1}, \mathbf{z}_k), \quad V' = \text{span}(\mathbf{z}_k, \mathbf{z}_{k+1}).$$

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